Homomorphism graphs and Descriptive combinatorics

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Descriptive Dynamics and Combinatorics seminar
8.11.2021

Joint work with Sebastian Brandt, Yi-Jun Chang, Christoph Grunau, Václav Rozhoň and Zoltán Vidnyánszky, should appear on arXiv tomorrow.

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The motivation comes from the adaptation of Marks' method to the LOCAL model of distributed computing, which was itself motivated by recent results of Bernshteyn.

Rather curiously, this adaptation gives a better insight back in descriptive combinatorics.

At the end of this talk we will see a new proof of the following result of Conley, Jackson, Marks, Seward, and Tucker-Drob.

Theorem (CJMST-D)

For each $\Delta > 2$, there is an acyclic Δ-regular hyperfinite Borel graph $\mathcal{G}$ such that $\chi_{\mathcal{B}}(\mathcal{G}) = \Delta + 1$.

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It is easy to see that $\mathcal{G}$ is Δ-regular, does not contain loops, but it might contain cycles or multiple edges.

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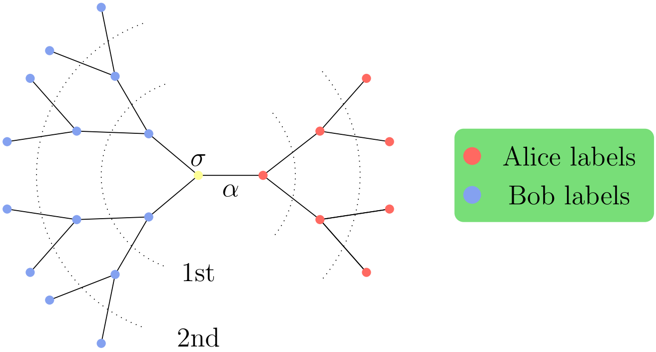

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By pigeonhole principle, there are $\ell_0 \neq \ell_1 \in \mathbb{N}$ and $i \in \Delta$ so that Bob has winning strategy for both $\mathbb{G}(\ell_0, i)$ and $\mathbb{G}(\ell_1, i)$.

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- each computer runs the same algorithm,
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In order to make this non-trivial, we need to break symmetries!! Unique identifiers.

In another words, for every local rule $\mathcal{A}$ of locality $O(\log^* n)$ there is a finite tree T of size n with vertices labeled with unique identifiers from $\{1, \ldots, n\}$ such that $\mathcal{A}$ fails to produce Δ-coloring when applied on T.

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By the result of Bernshteyn, this follows also from the result of Marks.

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Why does this not work? Injectivity is NOT preserved under gluing strategies together!

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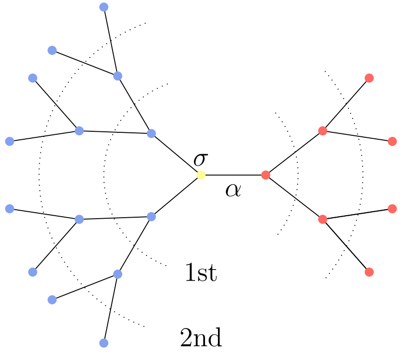

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- In the original construction, we used the pigeonhole principle.

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that has as a vertex set the set of all homomorphism from T_Δ to $\mathcal{H}$ (that preserve the edge labeling) and two homomorphism x, y are connected by an i-edge if moving the root along the i-edge transformers x to y.

The graph $\mathcal{H}$ is called the target graph and $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ is called the homomorphism graph.

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- $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ *is* Δ*-regular.*
- *If* $\chi_{\mathcal{B}}(\mathcal{H}) \leq \Delta$*, then* $\chi_{\mathcal{B}}(\mathbf{Hom}^e(T_\Delta, \mathcal{H})) \leq \Delta$*.*
- *If* $\mathcal{H}$ *is acyclic, then so is* $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$*.*

Basic results

Observation

Suppose that the graph $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ *is non-empty.*

- $\mathbf{Hom}^e(T_\Delta, \mathcal{H})$ *is* Δ*-regular.*
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For example, $el\chi_{wpr-\mathbf{\Delta}^1_2}(\mathcal{H}) > \Delta$ if the Baire edge-labeled chromatic number, $el\chi_{\mathrm{Baire}}(\mathcal{H})$, is bigger than Δ.

Application

Theorem (CJMST-D)

For each $\Delta > 2$, there is an acyclic Δ-regular hyperfinite Borel graph $\mathcal{G}$ such that $\chi_{\mathcal{B}}(\mathcal{G}) = \Delta + 1$.

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Taking $\mathcal{G} = \mathbf{Hom}^e(T_\Delta, \mathcal{H})$ works as required. □

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- applications for different local problems on trees?

THANK YOU